

ℓ^2 -homology and planar graphs

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Abstract

In his 1930 paper [7], Kuratowski categorized planar graphs, proving that a finite graph Γ is planar if and only if it does not contain a subgraph that is homeomorphic to K_5 , the complete graph on 5 vertices, or $K_{3,3}$, the complete bipartite graph on six vertices. This result is also attributed to Pontryagin ([6]). In their 2001 paper [4], Davis and Okun point out that the $K_{3,3}$ graph can be understood as the nerve of a right-angled Coxeter system and prove that this graph is not planar using results from ℓ^2 -homology. In this paper, we employ a similar method using results from [9] to prove K_5 is not planar.

1 Introduction

Let S be a finite set of generators. A *Coxeter matrix* on S is a symmetric $S \times S$ matrix $M = (m_{st})$ with entries in $\mathbb{N} \cup \{\infty\}$ such that each diagonal entry is 1 and each off diagonal entry is ≥ 2 . The matrix M gives a presentation of an associated *Coxeter group* W :

$$W = \langle S \mid (st)^{m_{st}} = 1, \text{ for each pair } (s, t) \text{ with } m_{st} \neq \infty \rangle. \quad (1.1)$$

The pair (W, S) is called a *Coxeter system*. Denote by L the nerve of (W, S) . L is a simplicial complex with vertex set S , the precise definition will be given in section 2. In several papers (e.g., [1], [2], and [3]), M. Davis describes a construction which associates to any Coxeter system (W, S) , a simplicial complex $\Sigma(W, S)$, or simply Σ when the Coxeter system is clear, on which W acts properly and cocompactly. The two salient features of Σ are that (1) it is contractible and (2) that it admits a cellulation under which the nerve of each vertex is L . It follows that if L is a triangulation of \mathbb{S}^{n-1} , Σ is an aspherical n -manifold. Hence, there is a variation of Singer's Conjecture, originally regarding the (reduced) ℓ^2 -homology of aspherical manifolds, for such Coxeter groups.

Singer's Conjecture for Coxeter groups 1.1. *Let (W, S) be a Coxeter group such that its nerve, L , is a triangulation of \mathbb{S}^{n-1} . Then $\mathcal{H}_i(\Sigma_L) = 0$ for all $i \neq \frac{n}{2}$.*

For details on ℓ^2 -homology theory, see [3], [4] and [5]. Conjecture 1.1 holds for elementary reasons in dimensions 1 and 2. In [4], Davis and Okun prove that if Conjecture 1.1 holds for *right-angled* Coxeter systems in dimension n , then it also holds in dimension $n + 1$. (Here, *right-angled* means generators either commute, or have no relation). They also prove directly that Conjecture 1.1 holds for right-angled systems in dimension 3, and thus also in dimension 4. This result also follows from work by Lott and Lück ([8]) and Thurston ([10]) regarding Haken Manifolds. In [9], the author proves that Conjecture 1.1 holds for arbitrary Coxeter systems with nerve \mathbb{S}^2 .

Also in [4], Davis and Okun use their low dimensional results to prove the following generalization of Conjecture 1.1.

Lemma 1.2. (Lemma 9.2.3, [4]) *Suppose (W, S) is a right-angled Coxeter system with nerve L , a flag triangulation of \mathbb{S}^2 . Let A be a full subcomplex of L . Then*

$$\mathcal{H}_i(W\Sigma_A) = 0 \text{ for } i > 1.$$

Here, Σ_A is the Davis complex associated with the Coxeter system (W_A, A^0) , where W_A is the subgroup of W generated by vertices in A , with nerve A . It is a subcomplex of Σ .

Lemma 1.2 is the key to what Davis and Okun call “a complicated proof of the classical fact that $K_{3,3}$ is not planar,” (See section 11.4.1, [4]). We outline that argument in Section 3. The purpose of this paper is employ a similar argument to prove that K_5 is not planar.

The key step for us is proving a result analogous to Lemma 1.2, but for subcomplexes of arbitrary Coxeter systems.

Main Theorem. (See Theorem 4.5) *Let (W, S) be a Coxeter system with nerve L , a triangulation of \mathbb{S}^2 . Let A be full subcomplex of L with right-angled complement. Then*

$$\mathcal{H}_i(W\Sigma_A) = 0 \text{ for } i > 1.$$

Here A having a “right-angled complement” means that for generators s and t , the Coxeter relation $m_{st} \neq 2$ nor ∞ implies that the vertices corresponding to s and t are both in A . From the Main Theorem, it follows that K_5 is not planar.

2 The Davis complex

Let (W, S) be a Coxeter system. Given a subset U of S , define W_U to be the subgroup of W generated by the elements of U . A subset T of S is *spherical* if W_T is a finite subgroup of W . In this case, we will also say that the subgroup W_T is spherical. Denote by \mathcal{S} the poset of spherical subsets of S , partially ordered by inclusion. Given a subset V of S , let $\mathcal{S}_{\geq V} := \{T \in \mathcal{S} | V \subseteq T\}$. Similar definitions exist for $<, >, \leq$. For any $w \in W$ and $T \in \mathcal{S}$, we call the coset wW_T a *spherical coset*. The poset of all spherical cosets we will denote by $W\mathcal{S}$.

Let $K = |\mathcal{S}|$, the geometric realization of the poset \mathcal{S} . It is a finite simplicial complex. Denote by $\Sigma(W, S)$, or simply Σ when the system is clear, the geometric realization of the poset WS . This is the Davis complex. The natural action of W on WS induces a simplicial action of W on Σ which is proper and cocompact. K includes naturally into Σ via the map induced by $T \rightarrow W_T$. So we view K as a subcomplex of Σ , and note that K is a strict fundamental domain for the action of W on Σ .

The poset $\mathcal{S}_{>\emptyset}$ is an abstract simplicial complex. This simply means that if $T \in \mathcal{S}_{>\emptyset}$ and T' is a nonempty subset of T , then $T' \in \mathcal{S}_{>\emptyset}$. Denote this simplicial complex by L and call it the *nerve* of (W, S) . The vertex set of L is S and a non-empty subset of vertices T spans a simplex of L if and only if T is spherical.

Define a labeling on the edges of L by the map $m : \text{Edge}(L) \rightarrow \{2, 3, \dots\}$, where $\{s, t\} \mapsto m_{st}$. This labeling accomplishes two things: (1) the Coxeter system (W, S) can be recovered (up to isomorphism) from L and (2) the 1-skeleton of L inherits a natural piecewise spherical structure in which the edge $\{s, t\}$ has length $\pi - \pi/m_{st}$. L is then a *metric flag* simplicial complex (see Definition [2, I.7.1]). This means that any finite set of vertices, which are pairwise connected by edges, spans a simplex of L if and only if it is possible to find some spherical simplex with the given edge lengths. In other words, L is “metrically determined by its 1-skeleton.”

Recall that a simplicial complex L is *flag* if every nonempty, finite set of vertices that are pairwise connected by edges spans a simplex of L . Thus, it is clear that any flag simplicial complex can correspond to the nerve of a right-angled Coxeter system. For the purpose of this paper, we will say that labeled (with integers ≥ 2) simplicial complexes are *metric flag* if they correspond to the labeled nerve of some Coxeter system. We then treat vertices of metric flag simplicial complexes as generators of a corresponding Coxeter system. Moreover, for a metric flag simplicial complex L , we write Σ_L to denote the associated Davis complex.

A cellulation of Σ by Coxeter cells. Σ has a coarser cell structure: its cellulation by “Coxeter cells.” (References include [2] and [4].) The features of the Coxeter cellulation are summarized by [2, Proposition 7.3.4]. We point out that under this cellulation the link of each vertex is L . It follows that if L is a triangulation of \mathbb{S}^{n-1} , then Σ is a topological n -manifold.

Full subcomplexes. Suppose A is a full subcomplex of L . Then A is the nerve for the subgroup generated by the vertex set of A . We will denote this subgroup by W_A . (This notation is natural since the vertex set of A corresponds to a subset of the generating set S .) Let \mathcal{S}_A denote the poset of the spherical subsets of W_A and let Σ_A denote the Davis complex associated to (W_A, A^0) . The inclusion $W_A \hookrightarrow W_L$ induces an inclusion of posets $W_A \mathcal{S}_A \hookrightarrow W_L \mathcal{S}_L$ and thus an inclusion of Σ_A as a subcomplex of Σ_L . Note that W_A acts on Σ_A and that if $w \in W_L - W_A$, then Σ_A and $w\Sigma_A$ are disjoint copies of Σ_A . Denote by

$W_L \Sigma_A$ the union of all translates of Σ_A in Σ_L .

3 Previous results in ℓ^2 -homology

Let L be a metric flag simplicial complex, and let A be a full subcomplex of L . The following notation will be used throughout.

$$\mathfrak{h}_i(L) := \mathcal{H}_i(\Sigma_L) \quad (3.1)$$

$$\mathfrak{h}_i(A) := \mathcal{H}_i(W_L \Sigma_A) \quad (3.2)$$

$$\beta_i(A) := \dim_{W_L}(\mathfrak{h}_i(A)). \quad (3.3)$$

Here $\dim_{W_L}(\mathfrak{h}_i(A))$ is the von Neumann dimension of the Hilbert W_L -module $W_L \Sigma_A$ and $\beta_i(A)$ is the i^{th} ℓ^2 -Betti number of $W_L \Sigma_A$. The notation in 3.2 and 3.3 will not lead to confusion since $\dim_{W_L}(W_L \Sigma_A) = \dim_{W_A}(\Sigma_A)$. (See [4] and [5]).

0-dimensional homology. Let Σ_A be the Davis complex constructed from a Coxeter system with nerve A , so W_A acts geometrically on Σ_A . The reduced ℓ^2 -homology groups of Σ_A can be identified with the subspace of *harmonic i -cycles* (see [5] or [4]). That is, $x \in \mathfrak{h}_i(A)$ is an i -cycle and i -cocycle. 0-dimensional cocycles of Σ_A must be constant on all vertices of Σ_A . It follows that if W_A is infinite, and therefore the 0-skeleton of Σ_A is infinite, $\beta_0(A) = 0$.

Singer Conjecture in dimensions 1 and 2. As mentioned in Section 1, Conjecture 1.1 is true in dimensions 1 and 2. Indeed, let L be \mathbb{S}^0 or \mathbb{S}^1 , the nerve of a Coxeter system (W, S) . Then W is infinite and so, as stated above, $\beta_0(L) = 0$. Poincaré duality then implies that the top-dimensional ℓ^2 -Betti numbers are also 0.

Orbihedral Euler Characteristic. Σ_L is a geometric W -complex. So there are only finite number of W -orbits of cells in Σ_L , and the order of each cell stabilizer is finite. The *orbihedral Euler characteristic* of $\Sigma_L/W = K$, denoted $\chi^{\text{orb}}(\Sigma_L/W)$, is the rational number defined by

$$\chi^{\text{orb}}(\Sigma_L/W) = \chi^{\text{orb}}(K) = \sum_{\sigma} \frac{(-1)^{\dim \sigma}}{|W_{\sigma}|}, \quad (3.4)$$

where the summation is over the simplices of K and $|W_{\sigma}|$ denotes the order of the stabilizer in W of σ . Then, if the dimension of L is $n - 1$, a standard argument (see [5]) proves Atiyah's formula:

$$\chi^{\text{orb}}(K) = \sum_{i=0}^n (-1)^i \beta_i(L). \quad (3.5)$$

Joins. If $L = L_1 * L_2$, the join of L_1 and L_2 , where each edge connecting a vertex of L_1 with a vertex of L_2 is labeled 2, we write $L = L *_2 L_2$ and then $W_L = W_{L_1} \times W_{L_2}$ and $\Sigma_L = \Sigma_{L_1} \times \Sigma_{L_2}$. We may then use Künneth formula to calculate the (reduced) ℓ^2 -homology of Σ_L , and the following equation from [4, Lemma 7.2.4] extends to our situation:

$$\beta_k(L_1 * L_2) = \sum_{i+j=k} \beta_i(L_1) \beta_j(L_2). \quad (3.6)$$

If $L = P *_2 L_2$, where P is one point, then we call L a *right-angled cone*. $\Sigma_P = [-1, 1]$, so there are no 1-cycles, and so $\beta_1(P) = 0$. But, $\chi^{\text{orb}}(\Sigma_P/W_P) = 1/2$ so by equation 3.5, $\beta_0(P) = 1/2$. Thus, in reference to the right-angled cone L , equation 3.6 implies that

$$\beta_i(L) = \frac{1}{2} \beta_i(L_2) \quad (3.7)$$

The $K_{3,3}$ case. Along with Lemma 1.2, the above gives us enough to prove that $K_{3,3}$ is not planar. Indeed, let P_3 denote 3 disjoint points. Then $K_{3,3} = P_3 *_2 P_3$ is the nerve of a right-angled Coxeter system. Since $W_{K_{3,3}}$ is infinite, so $\beta_0(K_{3,3}) = 0$, and equations 3.4 and 3.5 give us that $\beta_1(P_3) = 1/2$. It then follows from equation 3.6 that $\beta_2(K_{3,3}) = 1/4$. Thus, if $K_{3,3}$ were a planar graph, it could be embedded as a full-subcomplex of a flag triangulation of \mathbb{S}^2 , where each edge is labeled 2. This triangulation of \mathbb{S}^2 corresponds to the nerve of a right-angled Coxeter system. But this contradicts Lemma 1.2. For details on this proof see [4, Sections 8, 9 and 11].

4 The K_5 Graph

Let K_5 denote the complete graph on 5 vertices. The right-angled methods above cannot be applied to K_5 because, if the edges are labeled with 2's, then K_5 cannot be embedded as a full subcomplex of a metric flag triangulation of \mathbb{S}^2 . However, K_5 is metric flag if the edges are labeled with 3's. For if r, s and t are generators of a Coxeter system such that $m_{rs} = m_{st} = m_{rt} = 3$, then $\{r, s, t\}$ is not a spherical subset and this set does not span a 2-simplex in the nerve of the corresponding Coxeter system. This simple observation leads to the following definition.

Definition 4.1. We say a full subcomplex A of a metric flag simplicial complex L has a *right-angled complement* if the label on all edges not in A is 2.

The following two Lemmas will be used in the set-up and proof of our Main theorem.

Lemma 4.2. *Let L be a metric flag simplicial complex, $A \subseteq L$ a full subcomplex with a right-angled complement. Let B be a full subcomplex of L such that $A \subseteq B$ and let $v \in B - A$ be a vertex. Then B_v , the link of v in B , is a full subcomplex of L .*

Proof. Let T be a subset of vertices contained in B_v and the vertex set of a simplex σ of L . Then T defines a spherical subset of the corresponding Coxeter system. Since the of T are in B_v , v commutes with each vertex of T . Thus $T \cup \{v\}$ is a spherical subset and therefore σ is in B_v . \square

Lemma 4.3. *Let L be a metric flag triangulation of \mathbb{S}^1 , let A be a full subcomplex of L . Then $\beta_i(A) = 0$ for $i > 1$.*

Proof. Consider the long exact sequence of the pair $(\Sigma_L, W\Sigma_A)$:

$$0 \rightarrow \mathfrak{h}_2(A) \rightarrow \mathfrak{h}_2(L) \rightarrow \mathfrak{h}(L, A) \rightarrow \dots$$

Since Conjecture 1.1 is true in dimension 2, $\mathfrak{h}_2(L) = 0$ and exactness implies the result. \square

For convenience, we restate the relevant result from [9] needed to prove K_5 is non-planar.

Theorem 4.4. (See Corollary 4.4, [9]) *Let L be a metric flag triangulation of \mathbb{S}^2 . Then*

$$\mathfrak{h}_i(L) = 0 \text{ for all } i$$

We are now ready to prove our main theorem, analogous to Lemma 1.2.

Theorem 4.5. *Let L be a metric flag triangulation of \mathbb{S}^2 , $A \subseteq L$ a full subcomplex with right-angled complement. Then*

$$\beta_i(A) = 0 \text{ for } i > 1$$

Proof. Let B be a full subcomplex of L such that $A \subseteq B \subseteq L$. We induct on the number of vertices of $L - B$, the case $L = B$ given by Theorem 4.4. Assume $\mathfrak{h}_i(B) = 0$ for $i > 1$. Let v be a vertex of $B - A$ and set $B' = B - v$. Then $B = B' \cup C_2 B_v$ where B_v (by Lemma 4.2) and B' are full subcomplexes. We have the following Mayer-Vietoris Sequence:

$$\dots \rightarrow \mathfrak{h}_i(B_v) \rightarrow \mathfrak{h}_i(B') \oplus \mathfrak{h}_i(C_2 B_v) \rightarrow \mathfrak{h}_i(B) \rightarrow \dots$$

B_v is a full subcomplex of L_v , the link of v in L , a metric flag triangulation of \mathbb{S}^1 . So Lemma 4.3 implies $\mathfrak{h}_i(B_v) = 0$, for $i > 1$. Thus, by equation 3.7, $\mathfrak{h}_i(C_2 B_v) = 0$ for $i > 1$. It follows from exactness that $\mathfrak{h}_i(B') = 0$. \square

The above Theorem can be restated as follows, cf. [4, Theorem 11.4.1].

Theorem 4.6. *Let A be a metric flag complex of dimension ≤ 2 . Suppose A is planar (that is, it can be embedded as a subcomplex of the 2-sphere). Then*

$$\beta_2(A) = 0.$$

Proof. By Mayer-Vietoris, we may assume A is connected. Suppose A is piecewise linearly embedded in \mathbb{S}^2 . By introducing a new vertex in the interior of each complementary region, and coning off the boundary of each region labeling each new edge with 2, we obtain a metric flag triangulation of \mathbb{S}^2 in which every edge not in A is labeled 2, that is, A has a right-angled complement. The result follows from the proof of Theorem 4.5. \square

We are now ready to prove K_5 is not planar.

Corollary 4.7. *K_5 is not planar.*

Proof. Label each edge of K_5 with 3, and thus K_5 is a metric flag complex. In this case, $\chi^{\text{orb}}(K_5) = \frac{1}{6}$. Then Atiyah's formula, equation 3.5, and the fact that $\beta_0(K_5) = 0$ imply that $\beta_2(K_5) > 0$, contradicting Theorem 4.6. \square

References

- [1] M. W. Davis. Groups generated by reflections and aspherical manifolds not covered by Euclidean space. *Annals of Mathematics*, 117:293–294, 1983.
- [2] M. W. Davis. *The Geometry and Topology of Coxeter Groups*. Princeton University Press, Princeton, 2007.
- [3] M. W. Davis and G. Moussong. Notes on nonpositively curved polyhedra. Ohio State Mathematical Research Institute Preprints, 1999.
- [4] M. W. Davis and B. Okun. Vanishing theorems and conjectures for the ℓ^2 -homology of right-angled Coxeter groups. *Geometry & Topology*, 5:7–74, 2001.
- [5] B. Eckmann. Introduction to ℓ^2 -methods in topology: reduced ℓ^2 -homology, harmonic chains, ℓ^2 -beti numbers. *Israel Journal of Mathematics*, 117:183–219, 2000.
- [6] J. W. Kennedy, L. V. Quintas, and M. M. Syslo. The Theorem on Planar Graphs. *Historia Mathematica*, 12:356–368, 1985.
- [7] K. Kuratowski. Sur le problème des courbes gauches en Topologie. *Fundamenta Mathematicae*, 15:217–283, 1930.
- [8] J. Lott and W. Lück. ℓ^2 -topological invariants of 3-manifolds. *Invent. Math.*, 120:15–60, 1995.
- [9] T. A. Schroeder. Geometrization of 3-dimensional Coxeter orbifolds and Singer's conjecture. *Geometriae Dedicata*, 140(1):163ff, 2009. DOI number: 10.1007/s10711-008-9314-5.
- [10] W. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc.*, 6:357–381, 1982.